LAMINAR BOUNDARY LAYER ON A SPINNING
CONE AT SMALL ANGLES OF ATTACK

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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ABSTRACT

The problem of finding the laminar, compressible boundary layer flow over a rotating cone at small angles of attack is considered. The cone angle and free stream Mach number are such that the non-viscous flow is conical. Also a Prandtl number unity, and a linear viscosity-temperature relation are assumed. The "exact" solution for small angle of attack and small spin is obtained by a perturbation method including the first order interaction effects. For a slender cone the results are used to calculate the Magnus effects due to displacement thickness. As yet there are no experimental measurements which yield a conclusive test of the analytical results for Magnus force given here.

LIST OF SYMBOLS

A_i	=	coefficients in angle of attack expansion of non-viscous flow.
C	=	proportionality factor in μ - T relation.
C_{fx}, $C_{f\phi}$	=	dimensionless skin friction in x and ϕ directions.
C.P.	=	center of pressure of Magnus force.
C.M.	=	center of mass of cone.
d	=	diameter of cone.
F, G	=	generalized stream functions.
F_{ij}, G_{ij}	=	coefficients in expansion of F and G in α and κ.
K_1, J_1, J_2	=	coefficients in expression for Δ.
K_T, K_F	=	Magnus moment and force coefficients (ballistic notation).
ℓ	=	length of cone.
L	=	reference length, $C\bar{\mu} / \bar{\rho}\, \bar{u}$
M	=	Mach number
p	=	pressure
q	=	$1 + \left[(\gamma - 1) \bar{M}^2/2\right]$
r	=	radius of cross-section
$\bar{R}$	=	Reynolds number with fluid properties at reference condition and length indicated by subscript.
s	=	coordinate along free stream velocity direction
S	=	base area of cone
T	=	temperature
u, v, w	=	velocities along x, y, ϕ coordinate lines.
U	=	free stream velocity
x	=	coordinate along cone generators
y	=	coordinate normal to cone surface
α	=	angle of attack
γ	=	ratio of specific heats
Δ	=	displacement surface
η	=	dimensionless coordinate
Θ	=	cone half-angle
θ	=	$\sin\ \Theta$
κ	=	dimensionless coordinate $\omega r/\bar{u}'$
λ	=	dimensionless coordinate $\eta/2\xi^{1/2}$

μ	=	coefficient of viscosity
ν	=	$\omega d/U$
ξ	=	$x^3/3$
ρ	=	density
ϕ	=	coordinate angle around circumference of cone
ω	=	angular velocity of cone about axis
Ω	=	$\omega \ell \Theta / \bar{u}'$
$\bar{\ }$	=	indicates reference condition taken as conditions on surface of cone for zero angle of attack.
0	=	subscript indicating free stream value.
1	=	subscript indicating outer edge of boundary layer.
'	=	indicates quantity with dimensions, except in Eqs. (1) and (6) - (19).

INTRODUCTION

The study of three-dimensional boundary layer flows is becoming increasingly important. The class of flows for which "exact" solutions can be obtained is limited to flat plate and yawed cylinder type problems.[1] However, if one considers flows for which the departure from a basic two-dimensional flow is small, then by application of small perturbation methods a much wider class of flows becomes amenable to calculation.[1,2] It is interesting to note that the boundary layer flow over a rotating body of revolution at zero angle of attack can be classed roughly between the two-and-three-dimensional problems. For small relative spin, however, this problem is fairly easy to solve.[3] A body of revolution with non-zero angle of attack which is also rotating is a rather intricate three-dimensional problem but one which is of basic importance in ballistics.

To the author's knowledge only one example of this type of flow has been considered. Martin[4] has solved the case of a semi-infinite open-ended cylinder in incompressible flow. The main purpose of his work was to test a hypothesis that Magnus effects on slender bodies at small angles of attack are caused by the displacement effect of the boundary layer. Since the prediction of Magnus effects has remained rather elusive, the fact that Martin obtained some meaningful results was encouraging. However, because of the model chosen, experimental verification could not be conclusive. Thus it was decided to try a similar analysis for a more easily realizable model. Since it is desirable to take into account compressibility for ballistic applications, the cone in supersonic flow seems to be the obvious choice for an analytical solution including angle of attack and spin effects. This is, of course, because when the external flow is conical there is no pressure gradient along generators, and this allows a boundary layer solution to be obtained by introducing similarity variables.

The three special cases of the above problem have been worked out: for zero angle of attack and zero spin by Hantsche and Wendt;[5] for zero spin by Moore;[6] and for zero angle of attack by Illingworth.[3] Moore's

analysis is restricted to the first power in angle of attack, and Illingworth considers up to cubic terms in the spin. If one is interested in only first order effects of both angle of attack and spin, these two results can be superposed since the perturbations enter linearly, However, to obtain the interaction effects, which are necessary for Magnus effects, additional analysis must be made. This is what has been done here. Only laminar flow is considered; compressibility is, of course, not neglected; heat conduction is allowed for with the assumption of Prandtl number unity and a linear viscosity-temperature relationship. In the solution to the boundary layer problem the only other assumption (aside from the representation in terms of small perturbation) is that the external flow Mach number and the cone angle are such that the flow is conical. When the results are applied to Magnus force calculations slender body theory is used so that the restrictions to small cone angles and slightly supersonic Mach numbers are necessary.

Aside from the interest in Magnus effects the three-dimensional problem considered here has some intrinsic interest so that additional results are given, e.g., velocity profiles and skin friction. In particular, the following should be noted. To obtain generally useful results for three-dimensional flows approximate (momentum-integral) methods must be developed; this is especially true if the turbulent case is to be considered. Reliability of these methods cannot be based alone on experience in two-dimensional flows since there are fundamental differences in the two cases. Thus exact solutions are necessary to test proposed approximate methods.

The Magnus effects to be considered here are those commonly observed in ballistics. In the terminology of airplane dynamics the Magnus effects are a side force and yawing moments due to angle of attack which implies that the body must be spinning for non-zero Magnus effects. It is clear that non-viscous flow could not allow Magnus effects (this is discussed more fully in Reference 4) thus one is forced to investigate the boundary layer flow.

Assuming a Prandtl number equal to one and a linear viscosity-temperature relation give the boundary layer solution for no heat transfer at the surface only if the surface is stationary. For a spinning body this is not the case, as Illingworth[3] has noted. For the cone one finds that, exactly, the solution is for a surface temperature which varies as the square of the distance along a generator. An alternate interpretation is that the heat transfer at the surface is zero neglecting quadratic terms in α and $\mathcal{K}$. (Mr. Martin Fiebig called to the author's attention an error in the original manuscript concerning this point.)

EQUATIONS OF MOTION

The form of the equations of motion used here is taken from the work by Moore[8,7] and will only be summarized. A two-component vector potential is introduced to satisfy the continuity equation exactly (generalization of stream function). Prandtl number unity and the viscosity temperature relation

$$\mu / \bar{\mu} = CT / \bar{T}$$

are assumed, where the bar refers to the reference condition taken as the non-viscous flow at zero angle of attack evaluated at the surface and C is a suitably chosen constant. Two transformations are introduced; one like that of Howarth to try to remove compressibility effects and another like that of Mangler to try to remove curvature effects. Non-dimensional dependent variables are used with the reference condition denoted by a bar with the same meaning as above, and lengths are made dimensionless with the length $L = (C\bar{\mu}) / (\bar{\rho}\,\bar{u})$. Let u and w denote (non-dimensional) velocities along generators (x coordinate) and around the circumference (ϕ - coordinate); see Figure 1. If F and G denote the two components of the vector potential mentioned above then

$$u = F_\eta, \qquad w = G_\eta$$

where the subscript denotes differentiation and

$$\eta = x(\bar{p} / p)^{1/2} \int_0^y \rho dy$$

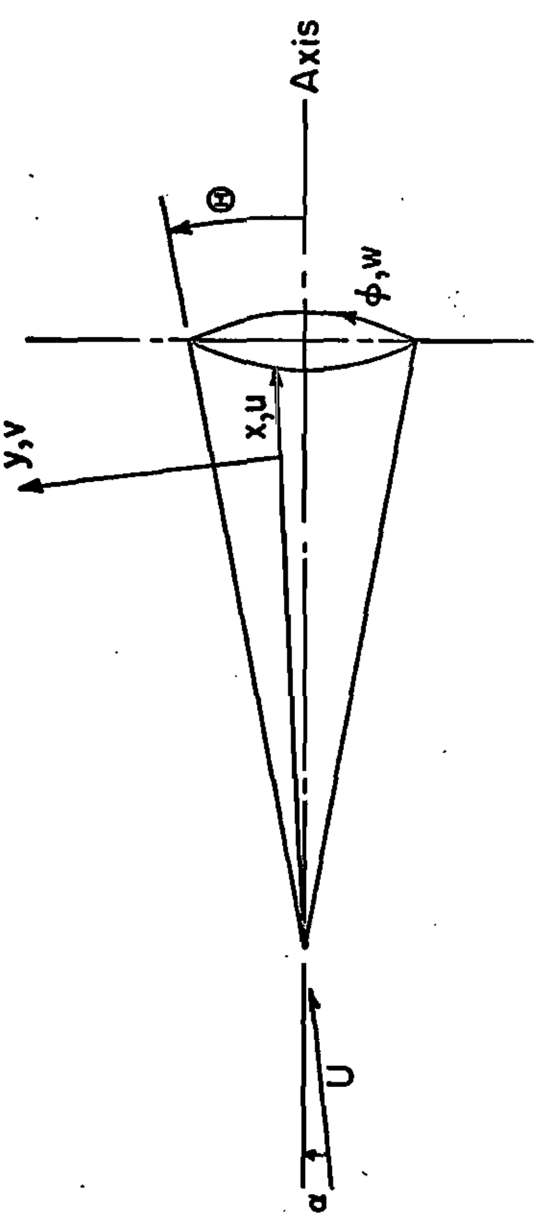

Figure I

Let Θ be the half-angle of the cone

$$\theta = \sin \Theta \ , \qquad \xi = x^3/3$$

then the equations of motion are (Equation 42, Reference 7).

$$F_\eta F_{\eta\xi} - \left[F_\xi + G_\phi/(3\theta\xi)\right] F_{\eta\eta} + G_\eta F_{\eta\phi}/(3\theta\xi) - (G_\eta)^2/(3\xi) = \tag{1a}$$

$$p'(\phi)\ GF_{\eta\eta}/(6\theta\xi p) + F_{\eta\eta\eta}$$

$$F_\eta G_{\eta\xi} - \left[F_\xi + G_\phi/(3\theta\xi)\right] G_{\eta\eta} + G_\eta G_{\eta\phi}/(3\theta\xi) + G_\eta F_\eta/(3\xi) = \tag{1b}$$

$$- p'(\phi)/(3\theta\xi\rho) + p'(\phi) GG_{\eta\eta}/(6\theta\xi p) + G_{\eta\eta\eta}$$

$$T + (F_\eta)^2 + (G_\eta)^2 = T_1 + u_1{}^2 + w_1{}^2 \tag{1c}$$

where p, ρ, and T are (non-dimensional) pressure, density, and temperature; $p'(\phi)$ is the derivative of p; and the subscript 1 denotes conditions at the outer edge of the boundary layer. The form of the variables η and ξ comes from the Howarth and Mangler transformations. For the boundary conditions at the outer edge of the boundary layer ($y = \eta = \infty$).

$$\begin{aligned} u &= u_1 = 1 - \alpha A_1 \cos \phi \\ w &= w_1 = \alpha A_2 \sin \phi \\ p/\overline{p} &= 1 + \alpha A_3 \cos \phi \\ \rho_1 &= 1 + \alpha A_4 \cos \phi \end{aligned} \tag{2}$$

since the flow is conical. The A_i are tabulated[9], see Reference 6 for further discussion. It is assumed that the rotation of the cone does not alter the external flow so that the pressure is given by the expression in (2). For the boundary conditions on the body surface ($y = \eta = 0$)

$$u = v = 0$$

$$w = \omega r \;/\; \overline{u}' \tag{3}$$

$$r = xL\theta$$

where ω is the angular velocity of the cone about its axis, r is the local radius and L is reference length defined above.

Since the pressure gradient is zero along the generators of the cone a Blasius type similarity variable is introduced.

$$\lambda = \eta/(2\sqrt{\xi}).$$

(This differs by a factor of one-half from the λ used by Moore.) However, because of the spin of the body there is no longer similarity in terms of λ. Thus variation with x must be allowed, and for this it is convenient to introduce a new variable κ ,

$$\kappa = \omega r \;/\; \overline{u'} = \omega L\theta \;(3\xi)^{1/3} \;/\; \overline{u}'$$

Furthermore the following forms for F and G are assumed:

$$F = \xi^{1/2}\, f(\kappa\,,\, \lambda,\, \phi)$$

$$G = \xi^{1/2}\, g(\kappa\,,\, \lambda,\, \phi)$$

For most ballistic applications the maximum value of κ will be less than 0.5, but it is assumed here that κ remains small enough so that f and g can be expanded in power series in κ . Hence, for this analysis, both the length of the cone and the angular velocity are limited. Thus we write

$$
\begin{aligned}
f &= f_0(\lambda,\, \phi) + \kappa\, f_1(\lambda,\, \phi) + \ldots\,. \\
g &= g_0(\lambda,\, \phi) + \kappa\, g_1(\lambda,\, \phi) + \ldots \\
T &= t_0(\lambda,\, \phi) + \kappa\, t_1(\lambda,\, \phi) + \ldots
\end{aligned}
\tag{4}
$$

For $\omega = 0$, f_0 and g_0 are the same as the f and g defined by Moore[6]. For $\alpha = 0$ all quantities are independent of ϕ, and we obtain essentially Illingworth's first order spin effect terms;[3] the correspondence is

$$(f_0)_I = f_0$$

$$(g_0)_I = g_{1_\lambda}/2$$

where the I indicates Illingworth's functions (f_1 would be identically zero for $\alpha = 0$).

The partial differential equations satisfied by the coefficient of κ in (4) will not be written down. In order to reduce the integration problem to one amenable to numerical processes the dependence of the coefficients on ϕ is assumed as follows (the form can be deduced from the appropriate boundary conditions):

$$\begin{aligned}
f_0 &= F_{00}(\lambda) - \alpha A_1 \cos\phi\, F_{01}(\lambda) + \dots \\
g_0 &= \alpha A_2 \sin\phi\, G_{01}(\lambda) + \dots \\
f_1 &= \alpha A_1 \sin\phi\, F_{11}(\lambda) + \dots \\
g_1 &= G_{10}(\lambda) - \alpha A_2 \cos\phi\, G_{11}(\lambda) + \dots
\end{aligned} \tag{5}$$

Only the functions of λ explicitly given in the expansions (5) will be obtained. If the assumed form of the solution be regarded as a power series expansion in the two small parameters α and κ it is evident that only one second order term is included, i.e., the product (interaction) term $\alpha\kappa$. The α^2 and κ^2 terms could be included at the expense of more numerical integrations. This may seem inconsistent, but, at least for Magnus effects, the second order terms are unimportant. The functions of λ indicated in (5) can be identified as follows; F_{00} governs the pure axial flow, F_{01} and G_{01} the angle of attack effects, G_{10} the spin effect, and F_{11} and G_{11} the interaction effects.

As expected F_{00} satisfies the Blasius equation. The remaining functions satisfy linear third-order ordinary differential equations with boundary conditions at $\lambda = 0$ and $\lambda = \infty$. These were not the equations that were integrated, however, and therefore will not be written down, for the following reason. These differential equations contain, through the A_1, the two parameters cone angle Θ and free stream Mach number M_0. But since they are linear it is an easy matter to rewrite the equations in a form which contains universal functions, i.e., independent of any parameters. Thus the following sequence of equations was obtained (the Blasius equation and that for G_{10}, which is related to Illingworth's g_0, require no change, but are rewritten in a different notation):

$$h_1''' + h_1 h_1'' = 0$$

$$h_1(0) = h_1'(0) = 0, \; h_1'(\infty) = 2 \qquad (6)$$

$$F_{00} = h_1$$

$$k_1''' + h_1 k_1'' - (2/3) h_1' k_1' = -(8/3)(1 - h_1'^2/4)$$

$$k_1(0) = k_1'(0) = 0, \qquad k_1'(\infty) = 0 \qquad (7)$$

Then $\quad G_{01} = h_1 + q\, k_1, \qquad q = 1 + (\gamma - 1)\overline{M}^2/2$

Define:

$$L_1(f) = f''' + h_1 f'' + h_1'' f$$

$$L_1(h_2) = 0 \qquad h_2(0) = h_2'(0) = 0 \qquad (8)$$

$$h_2'(\infty) = 1$$

$$L_1(h_3) - h_1'' k_1 = 0 \qquad h_3(0) = h_3'(0) = 0 \qquad (9)$$

$$h_3'(\infty) = 0$$

Then

$$F_{01} = \left[2A_2/(3\theta A_1)\right] h_1 + 2\left[1 - (2A_2/3\theta A_1)\right] h_2 + \left[2A_2\, q/(3\theta A_1)\right] h_3$$

Define:

$$L_2(f) = f''' + h_1 f'' - (4/3)\, h_1'\, f' \tag{10}$$

$$L_2(k_2) = 0 \quad k_2(0) = 0,\ k_2'(0) = 2, \qquad k_2'(\infty) = 0$$

Then

$$G_{10} = k_2$$

Also:

$$L_2(k_3) + k_2''\, h_2 - (4/3) k_2'\, h_2' = 0 \tag{11}$$

$$L_2(k_4) + k_2''\, (h_3 - k_1) + k_2' \left[k_1' - (4/3)\, h_3'\right] = 0 \tag{12}$$

$$L_2(k_5) - (1/3) h_1'\, k_2' = 0 \tag{13}$$

$$k_i(0) = k_i'(0) = k_i'(\infty) = 0, \qquad i = 3,4,5$$

Then

$$G_{11} = \left[2/(3\theta)\right] \left\{\left[2(3\theta A_1/\, 2A_2) - 1\right] k_3 + q k_4 + k_5\right\}$$

Define:

$$L_3(f) = f''' + h_1 f'' - (2/3) h_1'\, f' + (5/3)\, h_1''\, f$$

$$L_3(h_4) + (4/3)\, k_2'\, h_1' = 0 \tag{14}$$

$$L_3(h_5) + (4/3)\, k_2'\, k_1'' = 0 \tag{15}$$

$$L_3(h_6) - h_1'\, k_2' + h_1''\, k_5 = 0 \tag{16}$$

$$L_3(h_7) - k_2'\, h_3' + h_1''\, k_4 = 0 \tag{17}$$

$$L_3(h_8) - k_2'\, h_2' + h_1''\, k_3 = 0 \tag{18}$$

$$L_3(h_9) - k_2\ \ h_1'' = 0 \tag{19}$$

$$h_i(0) = h_i'(0) = h_i'(\infty) = 0, \qquad i = 4,5,\ .\ .\ .,\ 9.$$

Then

$$F_{11} = (A_2/A_1)(h_4 + qh_5) + (2/3\theta)^2 (A_2/A_1)(h_6 + qh_7) + (4/3\theta)(1 - 2A_2/3\theta A_1)h_8$$
$$+ (A_3/3\theta A_1)h_9$$

Equations (6) - (19) were integrated numerically on the ORDVAC and are tabulated in the Appendix. The functions were obtained to an accuracy of at least five significant figures. More accuracy was needed in the first few equations because of accumulation of errors. Equations (7), (8) and (9) are essentially the same as those integrated by Moore[6] (the numerical results agreed except that for the larger values of λ the agreement was only in the second or third significant figure). Equation (10) is essentially the same as one of Illingworth's (results agree to as many places as given). The remaining equations give the interaction effects.

RESULTS

After numerical integration of the sequence of ordinary differential equations it is possible to compute a number of quantities of interest.

(a) Velocity Profiles:

The non-dimensional velocity components u and w are obtained from

$$u = F_\eta = f_\lambda/2$$

$$w = G_\eta = g_\lambda/2$$

and making use of (4) and (5). For given free stream Mach number, M, and cone angle, Θ, the velocities are functions of α, κ, λ, and ϕ. In Fig. 2 u and w are plotted for the case $M = 1.82$, $\Theta = 10^\circ$, $\alpha = 1^\circ$, $\kappa = 0.1$ as functions of λ for $\phi = -\pi/2, 0, \pi/2$, and π.

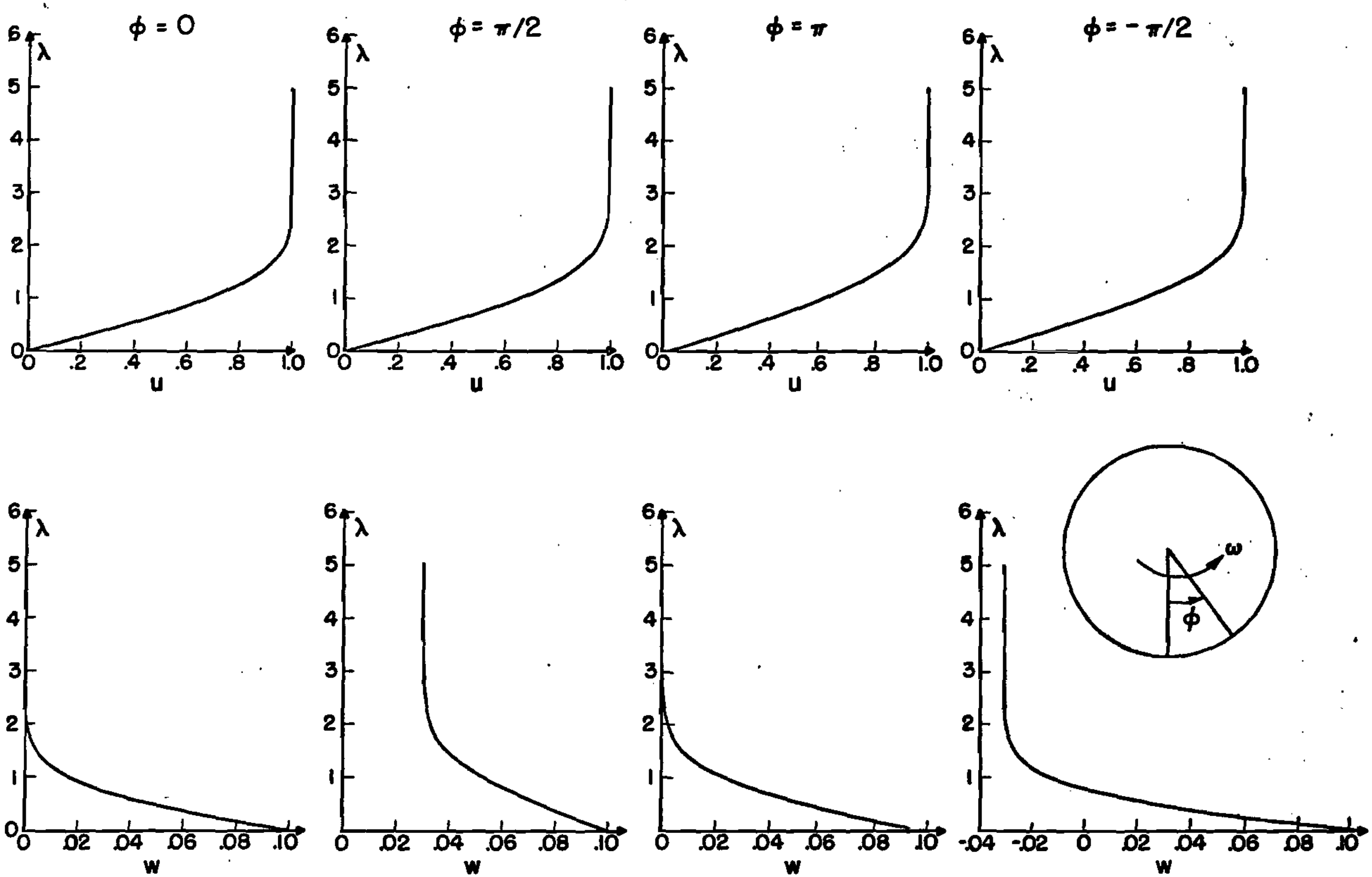

FIGURE 2
Non-dimensional velocity components u, w for $M = 1.82$, $\Theta = 10°$, $\alpha = 1°$, $\kappa = 0.1$

As expected there is a reversal in the w component for $\phi < 0$, i.e., where the rotational velocity opposes the direction of the flow outside the boundary layer. For the values of parameters chosen the u component does not change drastically around the circumference. There is no indication of separation type profiles. Note that it has not been necessary to compute the temperature variations. However this would have to be done if the velocity profiles were derived as functions of the physical coordinate y.

(b) Skin Friction:

The two components of skin friction, expressed in dimensionless form are

$$C_{fx} = 2\left[\mu' \partial u' / \partial y'\right]_{y' = 0} / \bar{\rho}' \bar{u}'^2$$

$$C_{f\phi} = 2\left[\mu' \partial w' / \partial y'\right]_{y' = 0} / \bar{\rho}' \bar{u}'^2$$

where the prime indicates that a quantity has dimensions. To the same order of approximation that has been carried throughout, the following expressions are obtained.

$$2C_{fx} = (3C/\bar{R}_x)^{1/2}\left[F_{00}''(0) - \alpha\left\{A_1F_{01}''(0) - (1/2)A_3F_{00}''(0)\right\}\cos\phi + \kappa\alpha A_1F_{11}''(0)\sin\phi\right] \tag{20}$$

$$2C_{f\phi} = (3C/\bar{R}_x)^{1/2}\left[\kappa G_{10}''(0) + \alpha A_2G_{01}''(0)\sin\phi + \kappa\alpha\left\{A_3G_{10}''(0)/2 - A_2G_{11}''(0)\right\}\cos\phi\right] \tag{21}$$

where $\bar{R}_x = x' \bar{\rho}' \bar{u}' / \bar{\mu}'$. The terms in (20) and (21) which are independent of or linear in α and κ can be obtained from References 6 and 3 respectively. The interaction terms $(\alpha\ \kappa)$ are

$$\partial^2 C_{fx}(\bar{R}_x/3C)^{1/2}/\partial\kappa\,\partial\alpha\,\sin\phi = A_1 F_{11}''(0)/2$$

$$= A_2\left[.6145 + .2522\,\bar{T}^{-1} + (2/3\theta)^2\,(.3592 + .1325\,\bar{T}^{-1})\right]$$

$$- (2/3\theta)\left[.5109\,A_1 + .2395\,A_3\right] \qquad (22)$$

$$\partial^2 C_{f\phi}(\bar{R}_x/3C)^{1/2}/\partial\kappa\,\partial\alpha\,\cos\phi = \left[A_3 G_{10}''(0) - 2A_2 G_{11}''(0)\right]/4$$

$$= .4896\,A_1 - .4898\,A_3 - (A_2/3\theta)\cdot\left[1.4496 + .5868\,\bar{T}^{-1}\right] \qquad (23)$$

where $\bar{T}^{-1} = (\gamma - 1)\bar{M}^2/2$. The right hand sides of equations (22) and (23) are plotted in Figures 3 and 4 respectively for cone angles of 10°, 12.5°, and 15°. According to the expression (20) for C_{fx} the effect of the $\alpha\kappa$ term, for negative ϕ, is to reduce the skin friction. This reduction is a maximum at $\phi = -\pi/2$. However, for resonably small values of α and κ the reduction is not enough to indicate component separation. For example to reduce C_{fx} to zero for $\Theta = 10^\circ$ at $\phi = -\pi/2$ it would be necessary to have $\alpha\kappa = .06$. For such values of α and κ the higher order terms would no longer be negligible. Since this is a three-dimensional flow it is by no means clear that component separation is pertinent to the general question of separation.

In Fig. 5 a sketch of $C_{f\phi}(\bar{R}_x/3C)^{1/2}$ is presented for $M = 1.82$ and $\Theta = 10^\circ$. It is seen that the variation of the skin friction is affected significantly by the interaction terms. However, the integrated value, which gives the spin reducing torque, is unaffected in the present approximation. (See Reference 3 for a discussion of the effect of the second order term in the spin on this torque.)

(c) Displacement Thickness

To find the displacement thickness for a three-dimensional boundary layer flow it is necessary to solve a first order partial differential equation[10,11]. For the cone problem if Δ is the displacement thickness

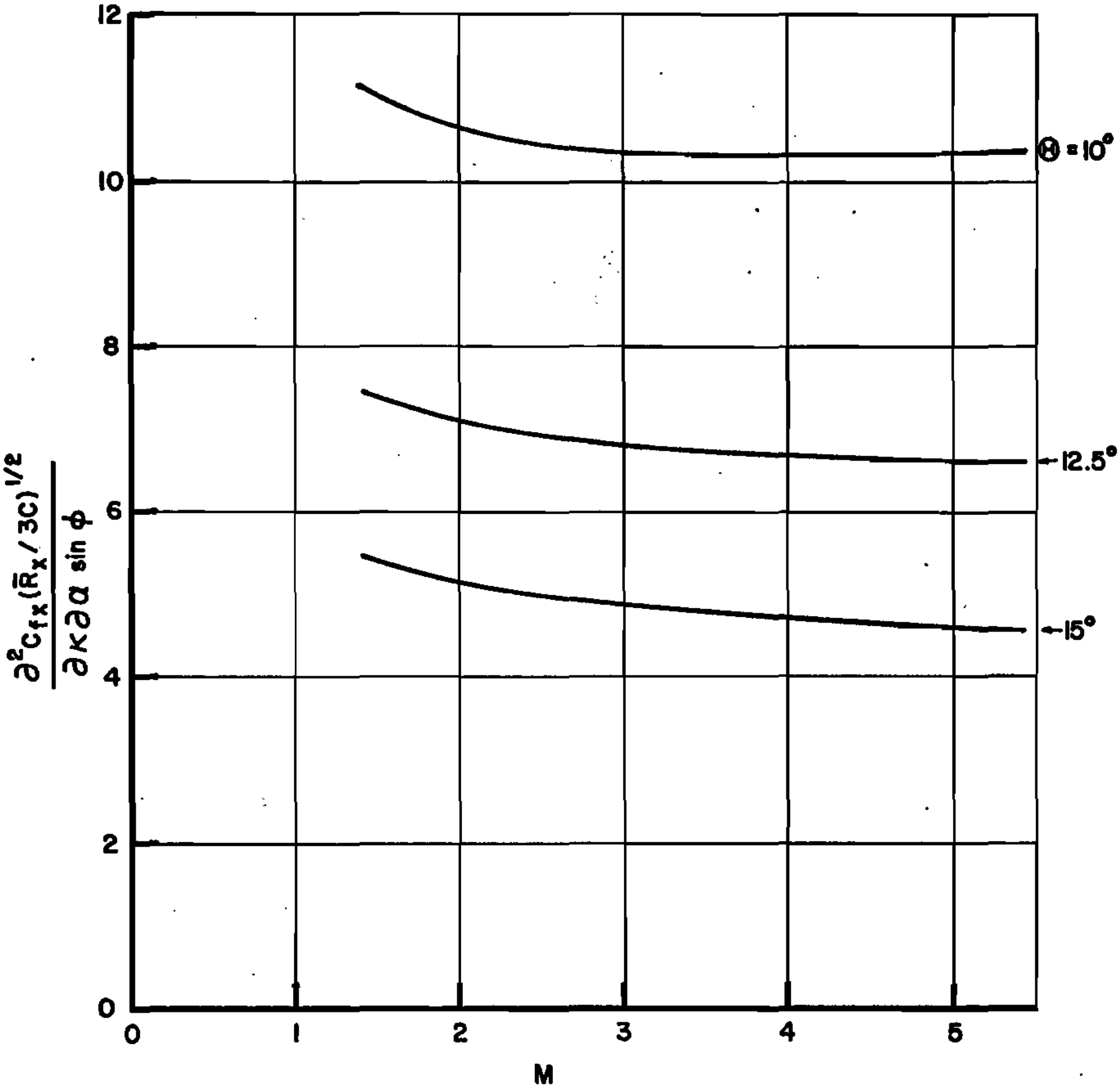

FIGURE 3

CONTRIBUTION OF INTERACTION TERM TO MERIDIONAL SKIN FRICTION, C_{fx}.

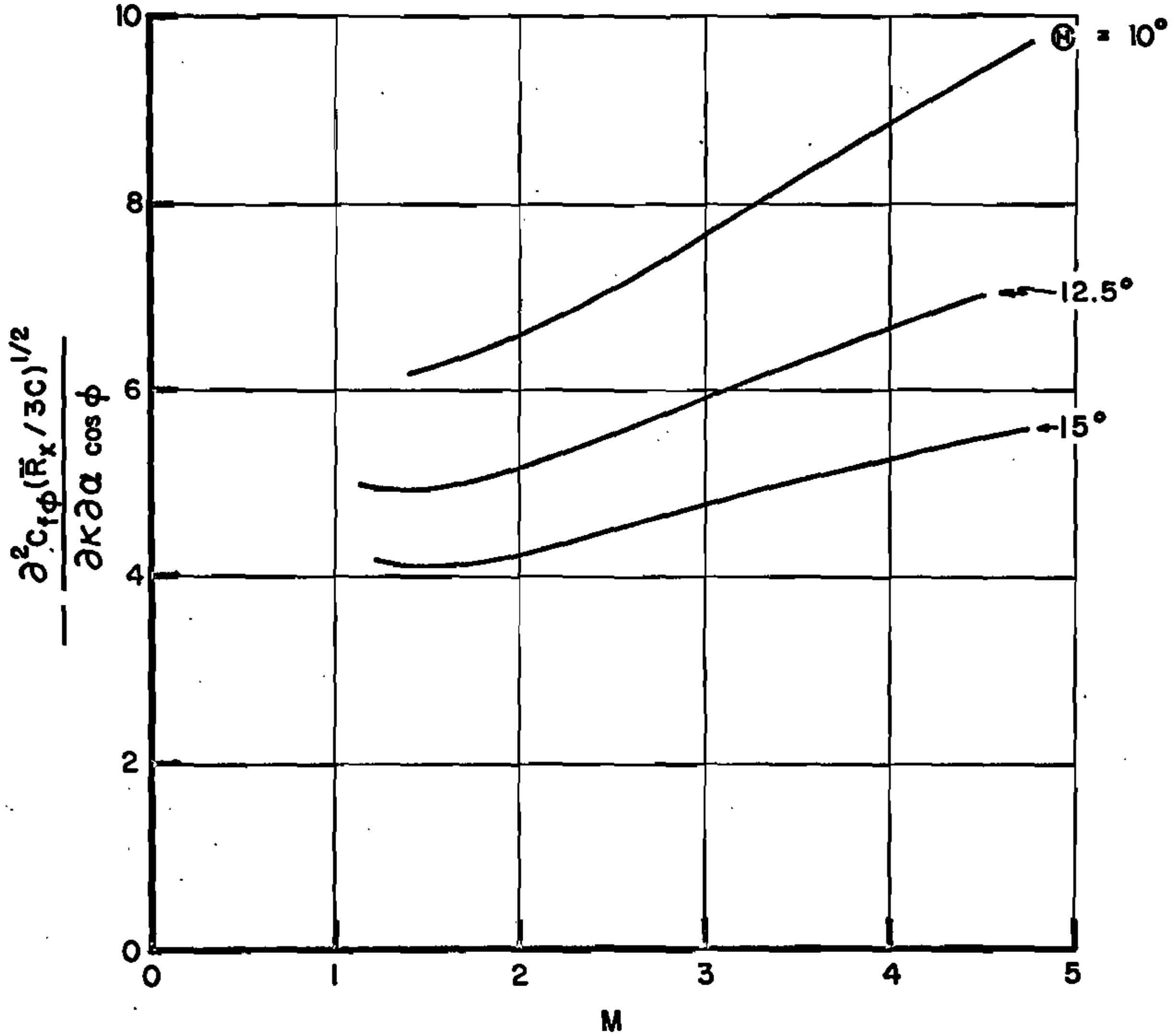

FIGURE 4

CONTRIBUTION OF INTERACTION TERM TO CIRCUMFERENTIAL SKIN FRICTION $C_{f\phi}$.

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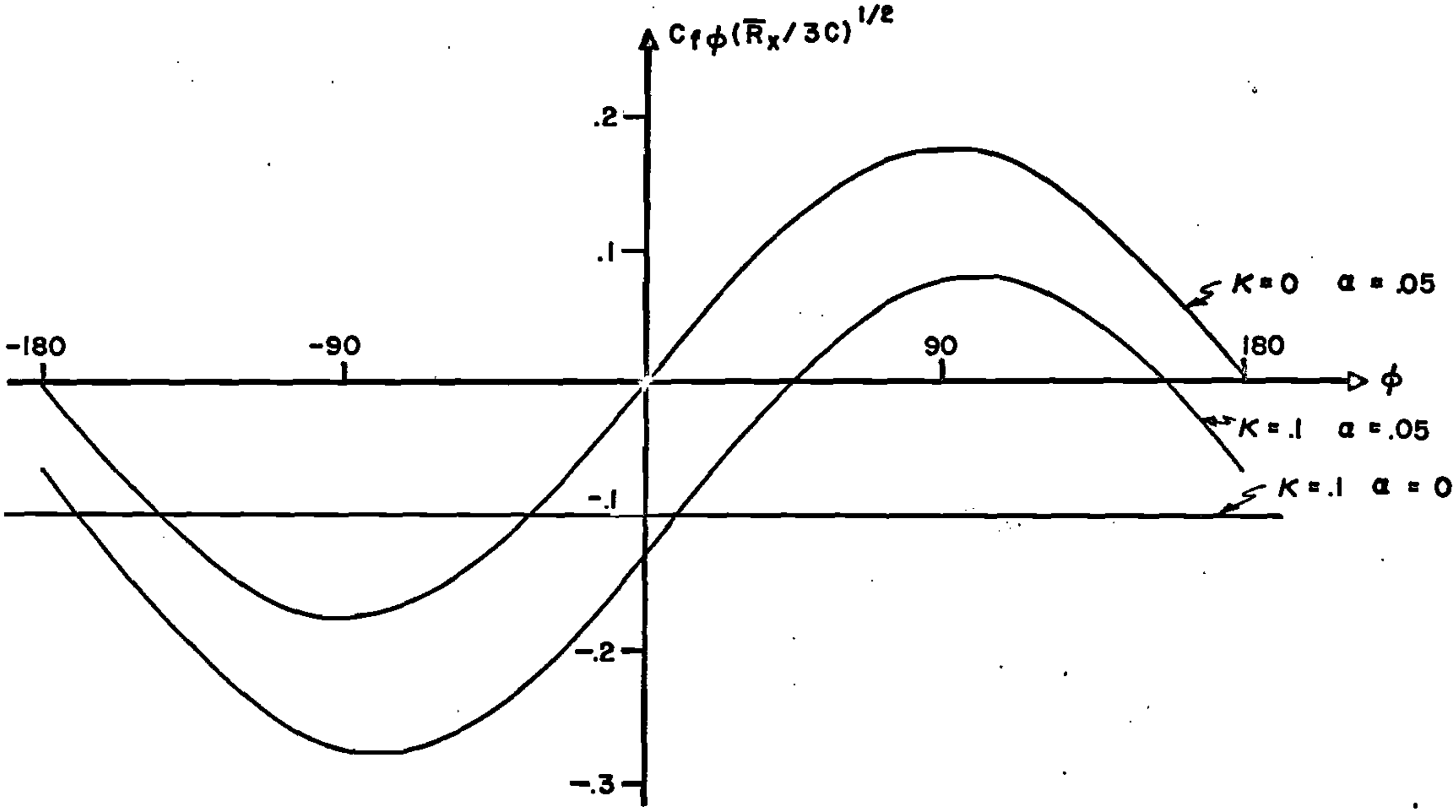

FIGURE 5

VARIATION OF $C_{f\phi}(\bar{R}_x/3C)^{1/2}$ AROUND CIRCUMFERENCE FOR M = 1.82 and Ⓗ = 10°.

$$\partial\left[\rho_1 u_1 x(\Delta - \delta_x)\right] /\partial x + \partial\left[\rho_1 w_1(\Delta - \delta_\phi)\right] /\theta\, \partial \phi = 0 \tag{24}$$

where $\delta_x = L \int_0^\infty \left[1 - (\rho u/\rho_1 u_1)\right] dy$

$$\delta_\phi = L \int_0^\infty \left[1 - (\rho w/\rho_1 w_1)\right] dy$$

$$L = C\, \overline{\mu} \,/\, \overline{\rho}\, \overline{u}$$

Integrating (24) and introducing a displacement thickness Reynolds number

$$R_\Delta = \overline{\rho}'\ \overline{u}'\Delta/\ \overline{\mu}'$$

the following is obtained

$$R_\Delta = 2C\left[K_1 + \alpha J_1 \cos\phi - \alpha\kappa J_2 \sin\phi\right](x/3)^{1/2} \tag{25}$$

where K_1, J_1, and J_2 are functions of free stream Mach number and cone angle. From References 10 and 6, J_1 and K_1 can be obtained and

$$J_2 = K_4 + K_5 + (K_6/5\theta)$$

where $K_4 = (2/3\theta)\left[A_1\ (.4142 - .5198\ \overline{T}^{\,-1}) - A_3\ (.2158 + .2771\ \overline{T}^{\,-1})\right]$

$$+ A_2\left[.4106 + .6396\ \overline{T}^{\,-1} + .1697\ \overline{T}^{\,-2}\right.$$

$$\left. + (2/3\theta)^2\ (.3186 + .5247\ \overline{T}^{\,-1} + .1495\ \overline{T}^{\,-2})\right]$$

$$K_5 = A_2\ \overline{T}^{\,-1}\ (.6692 + .2493\ \overline{T}^{\,-1})$$

$$K_6 = A_2\ (2/3\theta)(.9207 + 3289\ \overline{T}^{\,-1}) - .6546\ A_1 - .6538\ A_3$$

The $\alpha\kappa$ term in (25) makes the displacement thickness unsymmetrical with respect to the plane of yaw and is indicative of Magnus effects. For these effects, which will be considered in the next section, it is convenient to have $J_2 U//\overline{u}$ which is plotted in Fig. 6 where U is the free stream velocity. This velocity ratio is introduced because the dimensionless force and moment coefficients are conventionally based on free stream velocity.

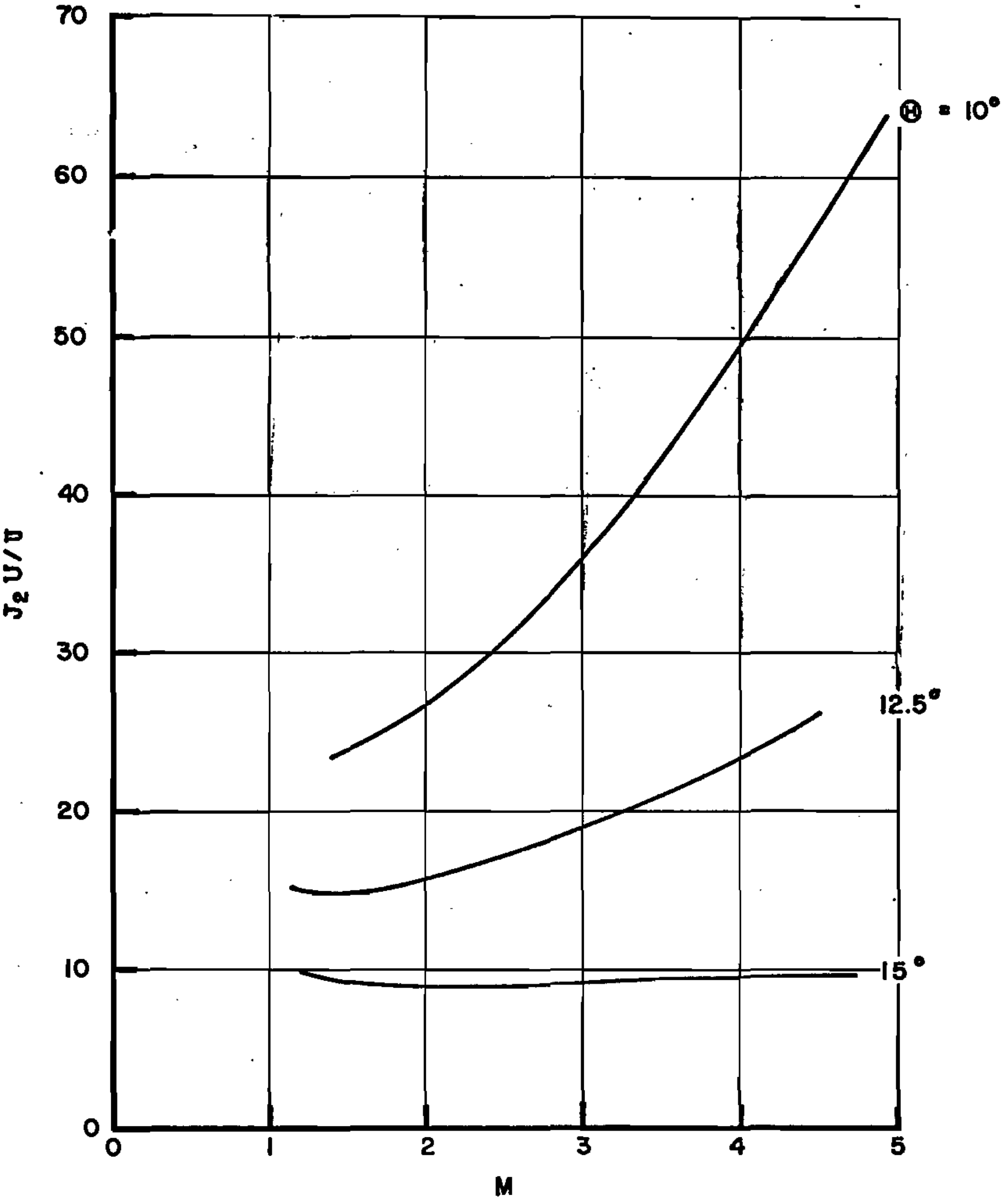

FIGURE 6

CONTRIBUTION OF INTERACTION TERM TO DISPLACEMENT THICKNESS Δ.

In Fig. 7 the variation of Δ around the circumference is shown for $M = 1.82$, $\Theta = 10^\circ$. The effect of the spin on displacement thickness is to rotate it in the direction of the spin. For the value of α and κ of Fig. 7 the maximum and minimum are rotated through 15° from the plane of yaw.

MAGNUS EFFECTS

The concept that the unsymmetrical displacement thickness on a rotating body at angle of attack causes Magnus effects was used by Martin[4] for the case of a semi-infinite cylinder. Basically the same idea will be used here. The potential flow over the distorted body, i.e., body plus displacement thickness, is calculated by means of slender body theory. Thus we now assume that the cone angle is small. The displacement thickness as given by (25) is measured normal to the body surface but for the approximation of slender body theory this thickness can be added to the cross-section of the body normal to its axis. Then the cross-section of the body can be expressed in polar coordinates as

$$r'/\ell = a + b \cos \phi + c \sin \phi \qquad (26)$$

where $a = (x' \Theta/\ell) + 2K_1(Cx'/3R_\ell \ell)^{1/2}$

$$b = 2\alpha\, J_1(Cx'/3R_\ell \ell)^{1/2}$$

$$c = -2\alpha\, J_2\Omega(x'/\ell)(Cx'/3R_\ell \ell)^{1/2}$$

$$\Omega = \omega\ \Theta\ \ell/\bar{u}'$$

with ℓ the length of the cone, R_ℓ the Reynolds number based on ℓ and the surface quantities at zero angle of attack, and the prime indicates a quantity with dimensions. To the first order in α the expression (26) for the cross-section is a circle of radius a and center (b, c).

The details of the slender body calculation need not be given since Ward[12] has presented all the necessary formulas for the cross force and moments. Ward uses, instead of the axis of the body, the s'-axis parallel to the direction of the free stream. With respect to the s'-axis the center of the circle (26) is (b*, c) where

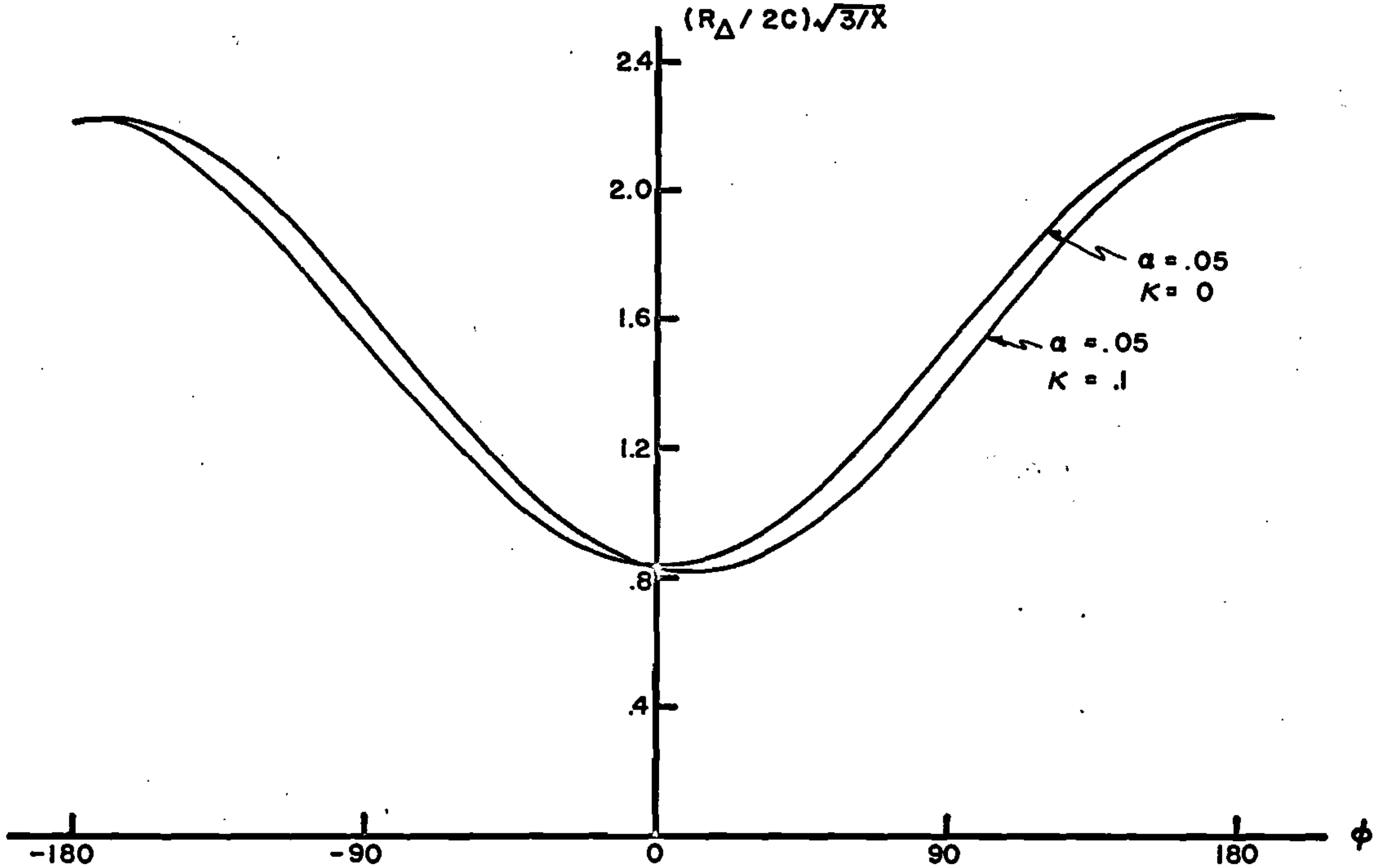

FIGURE 7

VARIATION OF $(R_\Delta/2C)(x/3)^{1/2}$ AROUND CIRCUMFERENCE FOR M = 1.82 and Ⓗ = 10°

$$b^* = b + \alpha s$$

and for the approximation made here $x' = s$. Then the complex transverse force, F, is given by

$$2F/\rho_o U^2 S = -2(db^*/ds + i\,dc/ds)_{s=1} \tag{27}$$

where S is the base area and ρ_o the free stream density. The real and imaginary parts give the lift and cross-force respectively. The negative sign in (27) results from the choice of axes in the cross-section. Thus the displacement thickness causes a decrease in the lift coefficient from the value 2α by an amount

$$2\alpha J_1 (C/3R_\ell)^{1/2}$$

(J_1 is negative). The Magnus force coefficient (in aerodynamic notation) is

$$\mathrm{Im}\left[2F/\rho_o U^2 S\right] = 6\,\alpha J_2 \Omega (C/3R_\ell)^{1/2} \tag{28}$$

The ratio of the Magnus force to lift force due to displacement thickness is

$$3\Omega\, J_2/J_1$$

which for $\Theta = 10^\circ$ is of the order of magnitude 10Ω which is less than one for the small spin contemplated here. Such a small Magnus force would be difficult to measure.

The center of pressure of the Magnus force is found to be

$$1 - \int_0^1 s^{1/2} \left[s\,\Theta + 2K_1 (Cs/3R_\ell)^{1/2}\right]^2 ds \Big/ \left[\Theta + 2K_1(C/3R_\ell)^{1/2}\right]^2$$

in units of the length of the cone. To within a few percent at most this result shows that the center of pressure is independent of Mach number and Reynolds number and is given by

$$\mathrm{C.P.} = (5/7)\ell.$$

In ballistic notation the Magnus force and moment coefficients, K_F and K_T respectively, are found to be

$$K_F = F/\ \rho_o U^2 d^2\ \nu\ \alpha = (\pi/8)(J_2 U/\overline{u'})(3C/R_\ell)^{1/2}$$

$$K_T = T/\ \rho_o U^2 d^3\ \nu\ \alpha = -(C.P._F - C.M.)K_F$$

where d is the diameter of the base, $\nu = \omega d/U$, $C.P._F$ is the center of pressure and C.M. is the center of mass both measured in calibers (diameters). The moment is taken about the center of mass. In Fig. 6 $J_2 U/\overline{u'}$ is plotted.

As yet there appear to be no experimental measurements which provide a conclusive test of these analytical results. There is some work in progress at the BRL which may yield such a test. Some measurements have been reported[13] giving only the Magnus moment on a slender cone. However, the boundary layer was definitely turbulent over most of the cone. Moreover, it is felt that the predicted moment will be more in error than the force. When the above result for K_T was compared with the results of Ref. 13 it was found to disagree considerably but in view of the above statements this is not regarded as conclusive.

Finally the contribution to the Magnus force of the meridional skin friction will be discussed. This was done for Martin's cylinder problem by Kelly[14] who found that skin friction contributed a negative Magnus force about 7 percent of displacement thickness contribution. Integrating (21) it is found that for the cone the $\alpha\,\kappa$ term gives a negative Magnus force which for $\Theta = 10^\circ$ is about 20 percent of that due to displacement thickness.

Martin[4] has attempted to extend his analysis of the laminar flow over a cylinder to the turbulent case by assuming that the Magnus force coefficient depends on the product of displacement thickness at the base for zero angle of attack and the length of the cylinder as in the laminar case. Aside from the fact that an empirical constant is left to be determined it is not clear that the above assumption is valid.

Thus the important matter of treating the turbulent boundary layer remains to be solved as well as the extension to more general bodies. This will necessitate the development of approximate methods for three-dimensional boundary layer flows.

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APPENDIX

Equation 6

λ	h_1	h_1'	h_1''				
.0	.00000	.00000	1.32822	2.5	3.28327	1.98308	.06362
.1	.00664	.13281	1.32793	2.6	3.48186	1.98849	.04536
.2	.02655	.26552	1.32587	2.7	3.68091	1.99231	.03171
.3	.05973	.39787	1.32031	2.8	3.88029	1.99495	.02172
.4	.10610	.52941	1.30955	2.9	4.07988	1.99675	.01459
.5	.16557	.65956	1.29202	3.0	4.27962	1.99794	.00960
.6	.23794	.78755	1.26635	3.1	4.47945	1.99872	.00620
.7	.32298	.91252	1.23146	3.2	4.67935	1.99922	.00392
.8	.42032	1.03351	1.18665	3.3	4.87929	1.99953	.00243
.9	.52951	1.14951	1.13172	3.4	5.07925	1.99972	.00147
1.0	.65002	1.25953	1.06700	3.5	5.27923	1.99984	.00088
1.1	.78119	1.36262	.99340	3.6	5.47922	1.99991	.00051
1.2	.92229	1.45796	.91236	3.7	5.67921	1.99995	.00029
1.3	1.07250	1.54490	.82581	3.8	5.87921	1.99997	.00016
1.4	1.23097	1.62301	.73602	3.9	6.07921	1.99998	.00009
1.5	1.39680	1.69208	.64544	4.0	6.27921	1.99999	.00004
1.6	1.56909	1.75216	.55651	4.1	6.47921	1.99999	.00002
1.7	1.74695	1.80352	.47150	4.2	6.67921	1.99999	.00001
1.8	1.92952	1.84665	.39234	4.3	6.87921	1.99999	.00000
1.9	2.11602	1.88223	.32050	4.4	7.07921	1.99999	.00000
2.0	2.30574	1.91103	.25693	4.5	7.27921	1.99999	.00000
2.1	2.49803	1.93391	.20207	4.6	7.47921	1.99999	.00000
2.2	2.69236	1.95174	.15589	4.7	7.67921	1.99999	.00000
2.3	2.88824	1.96536	.11793	4.8	7.87921	1.99999	.00000
2.4	3.08532	1.97557	.08748	4.9	8.07921	1.99999	.00000

Equation 7

λ	k_1	k_1'	k_1''				
.0	.00000	.00000	1.71521	2.5	.77131	.02106	- .07569
.1	.00813	.15820	1.44907	2.6	.77308	.01456	- .05514
.2	.03075	.28992	1.18625	2.7	.77429	.00987	- .03931
.3	.06525	.39568	.93035	2.8	.77510	.00657	- .02742
.4	.10905	.47635	.68521	2.9	.77563	.00428	- .01872
.5	.15972	.53321	.45479	3.0	.77598	.00274	- .01251
.6	.21496	.56793	.24298	3.1	.77620	.00172	- .00819
.7	.27264	.58255	.05336	3.2	.77634	.00106	- .00525
.8	.33088	.57945	- .11098	3.3	.77642	.00064	- .00329
.9	.38803	.56127	- .24774	3.4	.77647	.00038	- .00202
1.0	.44273	.53086	- .35559	3.5	.77650	.00022	- .00122
1.1	.49389	.49113	- .43429	3.6	.77652	.00012	- .00072
1.2	.54074	.44494	- .48478	3.7	.77653	.00007	- .00041
1.3	.58276	.39504	- .50915	3.8	.77653	.00003	- .00023
1.4	.61971	.34388	- .51046	3.9	.77654	.00002	- .00013
1.5	.65157	.29359	- .49254	4.0	.77654	.00001	- .00007
1.6	.67851	.24587	- .45967	4.1	.77654	.00000	- .00003
1.7	.70087	.20200	- .41629	4.2	.77654	.00000	- .00001
1.8	.71907	.16282	- .36664	4.3	.77654	.00000	- .00001
1.9	.73361	.12875	- .31453	4.4	.77654	.00000	- .00000
2.0	.74500	.09989	- .26311	4.5	.77654	.00000	- .00000
2.1	.75375	.07603	- .21481	4.6	.77654	.00000	- .00000
2.2	.76035	.05677	- .17128	4.7	.77654	.00000	- .00000
2.3	.76524	.04158	- .13344	4.8	.77654	.00000	- .00000
2.4	.76879	.02988	- .10163	4.9	.77654	.00000	- .00000

Equation 8

λ	h_2	h_2'	h_2''				
.0	.00000	.00000	.99624	2.5	2.05577	1.03002	-.07991
.1	.00498	.09961	.99572	2.6	2.15840	1.02272	-.06610
.2	.01991	.19905	.99244	2.7	2.26036	1.01678	-.05286
.3	.04477	.29792	.98378	2.8	2.36180	1.01210	-.04097
.4	.07945	.39554	.96729	2.9	2.46282	1.00853	-.03084
.5	.12381	.49104	.94076	3.0	2.56354	1.00587	-.02258
.6	.17756	.58330	.90245	3.1	2.66402	1.00395	-.01609
.7	.24032	.67110	.85128	3.2	2.76435	1.00260	-.01117
.8	.31158	.75312	.78697	3.3	2.86456	1.00167	-.00756
.9	.39071	.82808	.71026	3.4	2.96469	1.00105	-.00500
1.0	.47693	.89482	.62291	3.5	3.06478	1.00065	-.00322
1.1	.56937	.95240	.52766	3.6	3.16483	1.00039	-.00203
1.2	.66708	1.00021	.42808	3.7	3.26486	1.00023	-.00124
1.3	.76908	1.03801	.32825	3.8	3.36488	1.00013	-.00075
1.4	.87436	1.06599	.23238	3.9	3.46489	1.00007	-.00044
1.5	.98197	1.08475	.14438	4.0	3.56489	1.00004	-.00025
1.6	1.09103	1.09524	.06750	4.1	3.66490	1.00002	-.00014
1.7	1.20078	1.09870	.00399	4.2	3.76490	1.00001	-.00007
1.8	1.31059	1.09652	-.04499	4.3	3.86490	1.00000	-.00004
1.9	1.41995	1.09018	-.07952	4.4	3.96490	1.00000	-.00002
2.0	1.52853	1.08107	-.10064	4.5	4.06490	1.00000	-.00001
2.1	1.63611	1.07044	-.11019	4.6	4.16490	1.00000	-.00000
2.2	1.74260	1.05934	-.11047	4.7	4.26490	1.00000	-.00000
2.3	1.84799	1.04857	-.10398	4.8	4.36490	1.00000	-.00000
2.4	1.95235	1.03869	-.09310	4.9	4.46490	1.00000	-.00000

Equation 9

λ	h_3	h_3'	h_3''				
.0	.00000	.00000	-.38265	2.5	-.42381	-.02969	.09334
.1	-.00191	-.03825	-.38212	2.6	-.42635	-.02145	.07199
.2	-.00764	-.07632	-.37850	2.7	-.42816	-.01518	.05413
.3	-.01715	-.11375	-.36910	2.8	-.42944	-.01051	.03970
.4	-.03035	-.14987	-.35169	2.9	-.43031	-.00713	.02843
.5	-.04705	-.18377	-.32472	3.0	-.43090	-.00474	.01987
.6	-.06700	-.21447	-.28739	3.1	-.43128	-.00308	.01357
.7	-.08981	-.24091	-.23983	3.2	-.43153	-.00196	.00906
.8	-.11501	-.26213	-.18316	3.3	-.43169	-.00123	.00591
.9	-.14203	-.27731	-.11945	3.4	-.43179	-.00075	.00377
1.0	-.17025	-.28588	-.05162	3.5	-.43185	-.00045	.00235
1.1	-.19898	-.28761	.01683	3.6	-.43188	-.00026	.00143
1.2	-.22755	-.28262	.08215	3.7	-.43190	-.00015	.00085
1.3	-.25530	-.27140	.14072	3.8	-.43191	-.00008	.00050
1.4	-.28165	-.25480	.18948	3.9	-.43192	-.00004	.00028
1.5	-.30612	-.23391	.22619	4.0	-.43192	-.00002	.00016
1.6	-.32833	-.21000	.24972	4.1	-.43192	-.00001	.00008
1.7	-.34806	-.18441	.26007	4.2	-.43193	-.00000	.00004
1.8	-.36520	-.15839	.25827	4.3	-.43193	-.00000	.00002
1.9	-.37976	-.13309	.24621	4.4	-.43193	-.00000	.00001
2.0	-.39187	-.10941	.22630	4.5	-.43193	-.00000	.00000
2.1	-.40172	-.08801	.20115	4.6	-.43193	-.00000	.00000
2.2	-.40957	-.06927	.17330	4.7	-.43193	-.00000	.00000
2.3	-.41567	-.05336	.14497	4.8	-.43193	-.00000	.00000
2.4	-.42033	-.04023	.11791	4.9	-.43193	-.00000	.00000

Equation 10

λ	k_2	k_2'	k_2''	λ	k_2	k_2'	k_2''
.0	.00000	2.00000	-1.95932	2.5	1.30676	.00410	- .01758
.1	.19021	1.80464	-1.94228	2.6	1.30710	.00264	- .01180
.2	.36103	1.61257	-1.89422	2.7	1.30731	.00168	- .00778
.3	.51293	1.42668	-1.81951	2.8	1.30744	.00105	- .00503
.4	.64666	1.24941	-1.72262	2.9	1.30753	.00064	- .00320
.5	.76317	1.08275	-1.60808	3.0	1.30758	.00038	- .00200
.6	.86361	.92823	-1.48044	3.1	1.30761	.00023	- .00122
.7	.94926	.78695	-1.34415	3.2	1.30763	.00013	- .00074
.8	1.02147	.65954	-1.20350	3.3	1.30764	.00007	- .00043
.9	1.08164	.54626	-1.06247	3.4	1.30764	.00004	- .00025
1.0	1.13118	.44694	- .92465	3.5	1.30765	.00002	- .00014
1.1	1.17148	.36112	- .79308	3.6	1.30765	.00001	- .00008
1.2	1.20383	.28803	- .67024	3.7	1.30765	.00000	- .00004
1.3	1.22948	.22672	- .55797	3.8	1.30765	.00000	- .00002
1.4	1.24953	.17605	- .45744	3.9	1.30765	.00000	- .00001
1.5	1.26500	.13482	- .36924	4.0	1.30765	.00000	- .00000
1.6	1.27677	.10179	- .29338	4.1	1.30765	.00000	- .00000
1.7	1.28559	.07575	- .22940	4.2	1.30765	.00000	- .00000
1.8	1.29211	.05554	- .17648	4.3	1.30765	.00000	- .00000
1.9	1.29686	.04012	- .13356	4.4	1.30765	.00000	- .00000
2.0	1.30026	.02854	- .09941	4.5	1.30765	.00000	- .00000
2.1	1.30267	.01999	- .07275	4.6	1.30765	.00000	- .00000
2.2	1.30434	.01378	- .05235	4.7	1.30765	- .00000	- .00000
2.3	1.30548	.00935	- .03702	4.8	1.30765	- .00000	- .00000
2.4	1.30625	.00624	- .02573	4.9	1.30765	- .00000	- .00000

Equation 11

λ	k_3	k_3'	k_3''	λ	k_3	k_3'	k_3''
.0	.00000	.00000	- .48957	2.5	- .32439	- .01161	.04116
.1	- .00243	- .04853	- .47701	2.6	- .32537	- .00807	.03012
.2	- .00962	- .09466	- .44225	2.7	- .32604	- .00550	.02159
.3	- .02122	- .13639	- .38972	2.8	- .32650	- .00368	.01515
.4	- .03670	- .17216	- .32395	2.9	- .32680	- .00242	.01041
.5	- .05542	- .20089	- .24950	3.0	- .32699	- .00156	.00700
.6	- .07662	- .22192	- .17089	3.1	- .32712	- .00098	.00461
.7	- .09954	- .23507	- .09239	3.2	- .32720	- .00061	.00298
.8	- .12338	- .24054	- .01790	3.3	- .32725	- .00037	.00188
.9	- .14741	- .23890	.04928	3.4	- .32727	- .00022	.00116
1.0	- .17095	- .23101	.10657	3.5	- .32729	- .00013	.00071
1.1	- .19344	- .21797	.15225	3.6	- .32730	- .00007	.00042
1.2	- .21442	- .20098	.18548	3.7	- .32731	- .00004	.00024
1.3	- .23355	- .18129	.20629	3.8	- .32731	- .00002	.00014
1.4	- .25063	- .16011	.21550	3.9	- .32731	- .00001	.00007
1.5	- .26556	- .13853	.21454	4.0	- .32731	- .00000	.00004
1.6	- .27835	- .11747	.20524	4.1	- .32731	- .00000	.00002
1.7	- .28910	- .09768	.18969	4.2	- .32731	- .00000	.00001
1.8	- .29795	- .07968	.16995	4.3	- .32731	- .00000	.00000
1.9	- .30510	- .06377	.14797	4.4	- .32731	- .00000	.00000
2.0	- .31078	- .05010	.12541	4.5	- .32731	- .00000	.00000
2.1	- .31520	- .03866	.10360	4.6	- .32731	- .00000	.00000
2.2	- .31858	- .02932	.08349	4.7	- .32731	- .00000	.00000
2.3	- .32113	- .02189	.06567	4.8	- .32731	- .00000	.00000
2.4	- .32301	- .01610	.04959	4.9	- .32731	- .00000	.00000

Equation 12

λ	k_4	k'_4	k''_4				
.0	.00000	.00000	.58678	2.5	.32707	.00757	-.02835
.1	.00291	.05798	.56635	2.6	.32770	.00515	-.02029
.2	.01146	.11214	.51209	2.7	.32812	.00344	-.01421
.3	.02512	.15960	.43383	2.8	.32840	.00226	-.00975
.4	.04310	.19841	.34050	2.9	.32859	.00145	-.00655
.5	.06447	.22747	.24012	3.0	.32870	.00091	-.00431
.6	.08825	.24643	.13966	3.1	.32878	.00057	-.00278
.7	.11343	.25559	.04496	3.2	.32882	.00034	-.00175
.8	.13907	.25577	-.03941	3.3	.32885	.00020	-.00108
.9	.16433	.24816	-.11024	3.4	.32887	.00012	-.00066
1.0	.18850	.23424	-.16561	3.5	.32888	.00006	-.00039
1.1	.21102	.21558	-.20491	3.6	.32888	.00003	-.00022
1.2	.23151	.19377	-.22867	3.7	.32888	.00002	-.00013
1.3	.24972	.17031	-.23833	3.8	.32889	.00001	-.00007
1.4	.26556	.14650	-.23601	3.9	.32889	.00000	-.00004
1.5	.27905	.12342	-.22423	4.0	.32889	.00000	-.00002
1.6	.29030	.10189	-.20560	4.1	.32889	.00000	-.00001
1.7	.29950	.08244	-.18269	4.2	.32889	.00000	-.00000
1.8	.30687	.06541	-.15772	4.3	.32889	.00000	-.00000
1.9	.31266	.05090	-.13258	4.4	.32889	.00000	-.00000
2.0	.31713	.03886	-.10866	4.5	.32889	.00000	-.00000
2.1	.32051	.02910	-.08692	4.6	.32889	.00000	-.00000
2.2	.32302	.02138	-.06793	4.7	.32889	-.00000	-.00000
2.3	.32485	.01541	-.05190	4.8	.32889	-.00000	-.00000
2.4	.32615	.01090	-.03878	4.9	.32889	-.00000	-.00000

Equation 13

λ	k_5	k'_5	k''_5				
.0	.00000	.00000	-.11629	2.5	-.06250	-.00122	.00477
.1	-.00057	-.01148	-.11219	2.6	-.06260	-.00082	.00336
.2	-.00227	-.02221	-.10122	2.7	-.06267	-.00054	.00232
.3	-.00497	-.03157	-.08531	2.8	-.06271	-.00035	.00156
.4	-.00852	-.03917	-.06630	2.9	-.06274	-.00022	.00103
.5	-.01274	-.04478	-.04587	3.0	-.06276	-.00013	.00067
.6	-.01741	-.04834	-.02550	3.1	-.06277	-.00008	.00042
.7	-.02234	-.04992	-.00644	3.2	-.06278	-.00005	.00026
.8	-.02734	-.04971	.01036	3.3	-.06278	-.00002	.00016
.9	-.03223	-.04795	.02425	3.4	-.06278	-.00001	.00009
1.0	-.03689	-.04496	.03488	3.5	-.06278	-.00000	.00005
1.1	-.04120	-.04108	.04216	3.6	-.06278	-.00000	.00003
1.2	-.04509	-.03664	.04626	3.7	-.06279	-.00000	.00001
1.3	-.04852	-.03193	.04752	3.8	-.06279	-.00000	.00001
1.4	-.05147	-.02721	.04643	3.9	-.06279	-.00000	.00000
1.5	-.05397	-.02270	.04355	4.0	-.06279	-.00000	.00000
1.6	-.05602	-.01854	.03941	4.1	-.06279	-.00000	.00000
1.7	-.05769	-.01484	.03456	4.2	-.06279	-.00000	.00000
1.8	-.05901	-.01164	.02944	4.3	-.06279	-.00000	.00000
1.9	-.06004	-.00895	.02441	4.4	-.06279	.00000	.00000
2.0	-.06082	-.00674	.01972	4.5	-.06279	.00000	.00000
2.1	-.06140	-.00498	.01555	4.6	-.06279	.00000	.00000
2.2	-.06183	-.00361	.01197	4.7	-.06279	.00000	.00000
2.3	-.06213	-.00257	.00901	4.8	-.06279	.00000	.00000
2.4	-.06235	-.00179	.00663	4.9	-.06279	.00000	.00000

Equation 14

λ	h_4	h'_4	h''_4	λ	h_4	h'_4	h''_4
.0	.00000	.00000	.72468	2.5	.52433	.02210	-.07448
.1	.00360	.07190	.70791	2.6	.52620	.01562	-.05574
.2	.01427	.14059	.66139	2.7	.52751	.01083	-.04077
.3	.03153	.20337	.59078	2.8	.52841	.00736	-.02916
.4	.05468	.25813	.50176	2.9	.52902	.00490	-.02040
.5	.08283	.30330	.39995	3.0	.52942	.00320	-.01396
.6	.11498	.33788	.29088	3.1	.52967	.00205	-.00934
.7	.15004	.36141	.17982	3.2	.52984	.00128	-.00612
.8	.18690	.37394	.07172	3.3	.52994	.00079	-.00393
.9	.22448	.37600	-.02899	3.4	.53000	.00047	-.00246
1.0	.26178	.36851	-.11858	3.5	.53004	.00028	-.00151
1.1	.29791	.35275	-.19409	3.6	.53006	.00016	-.00091
1.2	.33211	.33023	-.25354	3.7	.53008	.00009	-.00053
1.3	.36379	.30261	-.29597	3.8	.53008	.00005	-.00031
1.4	.39252	.27160	-.32145	3.9	.53009	.00002	-.00017
1.5	.41805	.23885	-.33104	4.0	.53009	.00001	-.00009
1.6	.44028	.20586	-.32660	4.1	.53009	.00000	-.00005
1.7	.45926	.17392	-.31059	4.2	.53009	.00000	-.00002
1.8	.47514	.14403	-.28584	4.3	.53009	.00000	-.00001
1.9	.48816	.11694	-.25527	4.4	.53009	.00000	-.00000
2.0	.49864	.09308	-.22162	4.5	.53009	.00000	-.00000
2.1	.50689	.07264	-.18733	4.6	.53009	.00000	-.00000
2.2	.51328	.05557	-.15432	4.7	.53009	.00000	-.00000
2.3	.51811	.04168	-.12400	4.8	.53009	.00000	-.00000
2.4	.52171	.03065	-.09725	4.9	.53009	.00000	-.00000

Equation 15

λ	h_5	h'_5	h''_5	λ	h_5	h'_5	h''_5
.0	.00000	.00000	.50433	2.5	.28856	.01024	-.03434
.1	.00250	.04973	.48391	2.6	.28943	.00725	-.02571
.2	.00982	.09572	.43168	2.7	.29004	.00503	-.01884
.3	.02143	.13541	.35961	2.8	.29046	.00343	-.01350
.4	.03664	.16731	.27750	2.9	.29074	.00229	-.00947
.5	.05462	.19083	.19306	3.0	.29093	.00150	-.00650
.6	.07453	.20604	.11202	3.1	.29105	.00096	-.00436
.7	.09557	.21349	.03838	3.2	.29113	.00060	-.00287
.8	.11700	.21405	-.02535	3.3	.29118	.00037	-.00185
.9	.13818	.20879	-.07792	3.4	.29121	.00022	-.00116
1.0	.15860	.19885	-.11900	3.5	.29122	.00013	-.00071
1.1	.17784	.18536	-.14895	3.6	.29123	.00007	-.00043
1.2	.19559	.16940	-.16857	3.7	.29124	.00004	-.00025
1.3	.21167	.15195	-.17899	3.8	.29124	.00002	-.00014
1.4	.22596	.13387	-.18147	3.9	.29125	.00001	-.00008
1.5	.23845	.11587	-.17738	4.0	.29125	.00000	-.00004
1.6	.24916	.09856	-.16809	4.1	.29125	.00000	-.00002
1.7	.25820	.08238	-.15494	4.2	.29125	.00000	-.00001
1.8	.26569	.06766	-.13921	4.3	.29125	.00000	-.00000
1.9	.27178	.05459	-.12208	4.4	.29125	.00000	-.00000
2.0	.27666	.04325	-.10457	4.5	.29125	.00000	-.00000
2.1	.28049	.03366	-.08754	4.6	.29125	.00000	-.00000
2.2	.28345	.02571	-.07164	4.7	.29125	.00000	-.00000
2.3	.28569	.01927	-.05733	4.8	.29125	.00000	-.00000
2.4	.28735	.01418	-.04486	4.9	.29125	-.00000	-.00000

Equation 16

λ	h_6	h'_6	h''_6				
.0	.00000	.00000	− .56841	2.5	− .42044	− .01840	.06165
.1	− .00283	− .05641	− .55580	2.6	− .42200	− .01303	.04625
.2	− .01120	− .11040	− .52067	2.7	− .42309	− .00904	.03391
.3	− .02476	− .15993	− .46711	2.8	− .42384	− .00616	.02430
.4	− .04298	− .20335	− .39921	2.9	− .42435	− .00411	.01703
.5	− .06519	− .23943	− .32107	3.0	− .42469	− .00269	.01168
.6	− .09060	− .26736	− .23678	3.1	− .42490	− .00172	.00783
.7	− .11837	− .28672	− .15032	3.2	− .42504	− .00108	.00514
.8	− .14765	− .29748	− .06546	3.3	− .42513	− .00066	.00330
.9	− .17759	− .29998	.01432	3.4	− .42518	− .00040	.00207
1.0	− .20740	− .29488	.08600	3.5	− .42521	− .00023	.00127
1.1	− .23635	− .28313	.14713	3.6	− .42523	− .00013	.00077
1.2	− .26384	− .26587	.19598	3.7	− .42524	− .00007	.00045
1.3	− .28938	− .24438	.23157	3.8	− .42525	− .00004	.00026
1.4	− .31262	− .22000	.25378	3.9	− .42525	− .00002	.00014
1.5	− .33333	− .19404	.26325	4.0	− .42525	− .00001	.00008
1.6	− .35142	− .16773	.26131	4.1	− .42525	− .00000	.00004
1.7	− .36690	− .14210	.24984	4.2	− .42525	− .00000	.00002
1.8	− .37989	− .11800	.23104	4.3	− .42525	− .00000	.00001
1.9	− .39057	− .09605	.20723	4.4	− .42525	− .00000	.00000
2.0	− .39919	− .07665	.18063	4.5	− .42525	− .00000	.00000
2.1	− .40600	− .05995	.15323	4.6	− .42525	− .00000	.00000
2.2	− .41127	− .04597	.12664	4.7	− .42525	− .00000	.00000
2.3	− .41528	− .03456	.10207	4.8	− .42525	− .00000	.00000
2.4	− .41826	− .02547	.08028	4.9	− .42525	− .00000	.00000

Equation 17

λ	h_7	h'_7	h''_7				
.0	.00000	.00000	.26501	2.5	.23051	.01279	− .04130
.1	.00132	.02637	.26122	2.6	.23160	.00917	− .03148
.2	.00525	.05199	.25006	2.7	.23238	.00643	− .02342
.3	.01167	.07615	.23191	2.8	.23292	.00443	− .01702
.4	.02040	.09816	.20733	2.9	.23328	.00298	− .01208
.5	.03121	.11743	.17710	3.0	.23353	.00197	− .00838
.6	.04378	.13343	.14228	3.1	.23369	.00127	− .00568
.7	.05778	.14577	.10416	3.2	.23379	.00081	− .00377
.8	.07281	.15420	.06424	3.3	.23386	.00050	− .00244
.9	.08848	.15862	.02418	3.4	.23390	.00030	− .00155
1.0	.10440	.15909	− .01431	3.5	.23392	.00018	− .00096
1.1	.12018	.15586	− .04962	3.6	.23393	.00010	− .00058
1.2	.13546	.14932	− .08029	3.7	.23394	.00006	− .00034
1.3	.14995	.14000	− .10518	3.8	.23395	.00003	− .00020
1.4	.16339	.12850	− .12355	3.9	.23395	.00001	− .00011
1.5	.17560	.11551	− .13509	4.0	.23395	.00001	− .00006
1.6	.18647	.10170	− .13999	4.1	.23395	.00000	− .00003
1.7	.19594	.08772	− .13884	4.2	.23395	.00000	− .00001
1.8	.20403	.07411	− .13257	4.3	.23395	.00000	− .00000
1.9	.21079	.06133	− .12236	4.4	.23395	.00000	− .00000
2.0	.21633	.04972	− .10945	4.5	.23395	.00000	− .00000
2.1	.22078	.03949	− .09506	4.6	.23395	.00000	− .00000
2.2	.22428	.03072	− .08028	4.7	.23395	.00000	− .00000
2.3	.22698	.02342	− .06600	4.8	.23395	.00000	− .00000
2.4	.22901	.01748	− .05287	4.9	.23395	.00000	− .00000

Equation 18

λ	h_8	h'_8	h''_8	λ	h_8	h'_8	h''_8
.0	.00000	.00000	-.51085	2.5	-.40874	-.02036	.06682
.1	-.00254	-.05076	-.50128	2.6	-.41047	-.01452	.05058
.2	-.01009	-.09967	-.47415	2.7	-.41169	-.01015	.03739
.3	-.02236	-.14509	-.43191	2.8	-.41254	-.00695	.02701
.4	-.03894	-.18563	-.37711	2.9	-.41311	-.00466	.01907
.5	-.05929	-.22018	-.31249	3.0	-.41349	-.00307	.01316
.6	-.08275	-.24790	-.24093	3.1	-.41374	-.00198	.00889
.7	-.10862	-.26824	-.16545	3.2	-.41390	-.00125	.00587
.8	-.13615	-.28096	-.08918	3.3	-.41400	-.00077	.00379
.9	-.16456	-.28615	-.01521	3.4	-.41406	-.00047	.00239
1.0	-.19314	-.28418	.05354	3.5	-.41410	-.00028	.00148
1.1	-.22118	-.27570	.11446	3.6	-.41412	-.00016	.00089
1.2	-.24809	-.26162	.16542	3.7	-.41413	-.00009	.00053
1.3	-.27336	-.24300	.20491	3.8	-.41414	-.00005	.00030
1.4	-.29658	-.22104	.23217	3.9	-.41414	-.00002	.00017
1.5	-.31750	-.19697	.24724	4.0	-.41415	-.00001	.00009
1.6	-.33595	-.17198	.25087	4.1	-.41415	-.00000	.00005
1.7	-.35190	-.14713	.24449	4.2	-.41415	-.00000	.00002
1.8	-.36541	-.12335	.22997	4.3	-.41415	-.00000	.00001
1.9	-.37663	-.10134	.20946	4.4	-.41415	-.00000	.00000
2.0	-.38576	-.08158	.18515	4.5	-.41415	-.00000	.00000
2.1	-.39303	-.06436	.15911	4.6	-.41415	-.00000	.00000
2.2	-.39872	-.04976	.13308	4.7	-.41415	-.00000	.00000
2.3	-.40307	-.03770	.10846	4.8	-.41415	-.00000	.00000
2.4	-.40634	-.02799	.08618	4.9	-.41415	-.00000	.00000

Equation 19

λ	h_9	h'_9	h''_9	λ	h_9	h'_9	h''_9
.0	.00000	.00000	-.95816	2.5	-.85027	-.04839	.15585
.1	-.00478	-.09537	-.94503	2.6	-.85440	-.03471	.11892
.2	-.01898	-.18815	-.90631	2.7	-.85733	-.02439	.08857
.3	-.04224	-.27582	-.84322	2.8	-.85937	-.01679	.06441
.4	-.07390	-.35604	-.75751	2.9	-.86076	-.01132	.04576
.5	-.11312	-.42665	-.65159	3.0	-.86169	-.00748	.03177
.6	-.15885	-.48579	-.52876	3.1	-.86229	-.00485	.02156
.7	-.20985	-.53198	-.39320	3.2	-.86268	-.00307	.01431
.8	-.26478	-.56418	-.24996	3.3	-.86293	-.00191	.00929
.9	-.32221	-.58190	-.10477	3.4	-.86308	-.00116	.00589
1.0	-.38068	-.58527	.03625	3.5	-.86317	-.00069	.00366
1.1	-.43881	-.57499	.16704	3.6	-.86323	-.00040	.00222
1.2	-.49527	-.55239	.28201	3.7	-.86326	-.00023	.00132
1.3	-.54893	-.51927	.37661	3.8	-.86327	-.00013	.00077
1.4	-.59885	-.47785	.44765	3.9	-.86328	-.00007	.00044
1.5	-.64431	-.43057	.49365	4.0	-.86329	-.00003	.00024
1.6	-.68486	-.37995	.51490	4.1	-.86329	-.00002	.00013
1.7	-.72027	-.32835	.51335	4.2	-.86329	-.00001	.00007
1.8	-.75057	-.27793	.49230	4.3	-.86330	-.00000	.00003
1.9	-.77595	-.23040	.45600	4.4	-.86330	-.00000	.00001
2.0	-.79679	-.18708	.40910	4.5	-.86330	-.00000	.00000
2.1	-.81354	-.14878	.35621	4.6	-.86330	-.00000	.00000
2.2	-.82673	-.11590	.30149	4.7	-.86330	-.00000	.00000
2.3	-.83690	-.08843	.24833	4.8	-.86330	-.00000	.00000
2.4	-.84458	-.06609	.19925	4.9	-.86330	-.00000	.00000